Minimize Cost and Maximize Profit

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Let $C(x)$ be the cost of producing x units of commodity. Part of the cost is independent of the output level x, and is called **fixed cost**. The rest of the cost is called **variable cost** which depends on x. Hence, total cost is the sum of fixed cost, F, and variable cost, $C_v(x)$,

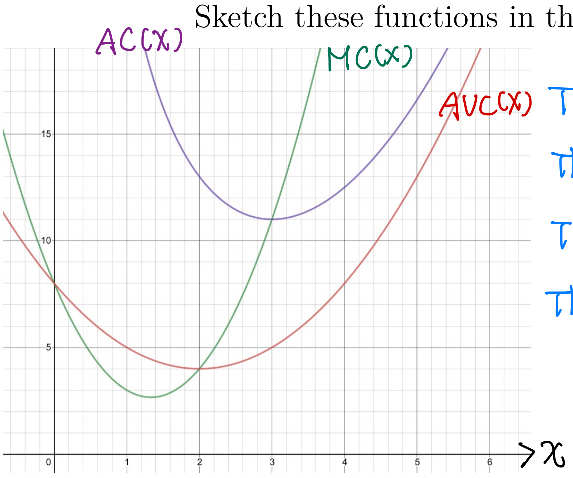
$$C(x) = F + C_v(x) \text{ where we assume that } F = C(0).$$

- The **average cost function** is $AC(x) = \dfrac{C(x)}{x}$.
- The **average variable cost function** is $AVC(x) = \dfrac{C_v(x)}{x}$.

A firm may want to minimize average cost or average variable cost.

Exercise: Suppose that the cost function is $C(x) = x^3 - 4x^2 + 8x + 18$.

1. Derive the marginal cost function, average cost function, and average variable cost function. Sketch these functions in the same figure.



The marginal cost: $MC(x) = C'(x) = 3x^2 - 8x + 8$.

The average cost: $AC(x) = \frac{C(x)}{x} = x^2 - 4x + 8 + \frac{18}{x}$.

The variable cost: $C_v(x) = C(x) - C(0) = x^3 - 4x^2 + 8x$.

The average variable cost: $AVC(x) = \frac{C_v(x)}{x} = x^2 - 4x + 8$.

2. At what output level is average cost minimized? Where does the marginal cost function intersect the average cost function?

① $\frac{d}{dx}AC(x) = 2x - 4 - \frac{18}{x^2} = \frac{2}{x^2}(x^3 - 2x^2 - 9) = \frac{2}{x^2}(x-3)(x^2+x+3)$

$AC'(x) < 0$ for $x \in (0,3)$, and $AC'(x) > 0$ for $x \in (3,\infty)$.

Hence $AC(x)$ obtains absolute minimum at $x = 3$.

② Solve $AC(x) = MC(x) \Rightarrow x^2 - 4x + 8 + \frac{18}{x} = 3x^2 - 8x + 8 \Rightarrow \frac{2}{x}(x^3 - 2x^2 - 9) = 0$

$\Rightarrow x = 3$.

3. At what output level is average variable cost minimized? Where does the marginal cost function intersect the average variable cost function?

① $\frac{d}{dx}AVC(x) = 2x - 4$. $\frac{d}{dx}AVC(x) < 0$ for $x \in (0,2)$ and

$\frac{d}{dx}AVC(x) > 0$ for $x \in (2,\infty)$. Hence $AVC(x)$ obtains absolute minimum at $x = 2$.

② Solve $MC(x) = AVC(x) \Rightarrow 3x^2 - 8x + 8 = x^2 - 4x + 8 \Rightarrow 2x^2 - 4x = 0$

$\Rightarrow \underline{x = 0}$ or 2. (Remark: $AVC(x) = \frac{C_v(x)}{x}$ is not defined at $x = 0$, but $\lim_{x \to 0} AVC(x)$ exists.)

4. Prove that the marginal cost function always passes through the minimum points of both the average cost function and the average variable cost function.

Suppose that the cost function $C(x)$ is differentiable. Then $AC(x) = \frac{C(x)}{x}$ and $AVC(x) = \frac{C(x)-C(0)}{x}$ are differentiable for $x>0$. If the average cost $AC(x)$ obtains minimum value at $x_0>0$, then by Fermat's Theorem, $AC'(x_0)=0$. However, $\frac{d}{dx}AC(x) = \frac{C'(x)x - C(x)}{x^2}$. $AC'(x_0)=0 \Rightarrow C'(x_0)x_0 = C(x_0)$

$\Rightarrow C'(x_0) = \frac{C(x_0)}{x_0}$ i.e. $MC(x_0) = AC(x_0)$.

If the average variable function obtains minimum value at $x_1>0$, then by Fermat's Theorem, $AVC'(x_1)=0$. Since $\frac{d}{dx}AVC = \frac{C'(x)\cdot x - C(x) + C(0)}{x^2}$, $AVC'(x_1)=0 \Rightarrow C'(x_1)x_1 - C(x_1) + C(0) = 0 \Rightarrow C'(x_1) = \frac{C(x_1)-C(0)}{x_1}$

i.e. $MC(x_1) = AVC(x_1)$.

(Remark: You can also show that $\lim_{x\to 0} AVC(x) = MC(0)$.)

Now we consider the **profit** of a firm, which is **total revenue minus total cost**. If $R(x)$ represents the revenue when x units are produced and sold, then profit of selling x units is $\Pi(x) = R(x) - C(x)$. It is assumed that the second derivatives of $R(x)$ and $C(x)$ exist and both are continuous. And usually a firm's goal is to maximize profit.

1. Show that if profit obtains maximum value at $x_0 > 0$, then $R'(x_0) = C'(x_0)$ and $R''(x_0) \le C''(x_0)$.

$R(x)$ and $C(x)$ are differentiable on $(0,\infty)$. If $\Pi(x) = R(x) - C(x)$ obtains maximum value at $x_0>0$, then $\Pi(x_0)$ is a local minimum and Π is differentiable at x_0. Fermat's Theorem tells us that $\Pi'(x_0) = R'(x_0) - C'(x_0) = 0$

$\Rightarrow R'(x_0) = C'(x_0)$.

Suppose that $R''(x_0) > C''(x_0)$. Then $\Pi''(x_0) > 0$ and by the second derivative test $\Pi(x_0)$ is a local minimum which contradicts to the fact that $\Pi(x_0)$ is maximum. Hence $R''(x_0) \le C''(x_0)$.

2. If the market is competitive, then the unit price of the product is constant, say, at p_0. Write down the revenue function $R(x)$ and show that the marginal cost is p_0 when profit is maximized.

$R(x) = P_0 \cdot x$.

When the profit is maximized at $x = x_0 > 0$, $R'(x_0) = C'(x_0)$.

However $R'(x_0) = P_0$. Therefore $C'(x_0) = P_0$ i.e. the marginal cost is P_0.

3. Suppose that when x units are demanded the unit price is $p = p(x) = -0.0032x + 10$ and total cost for producing x units is $C(x) = 4000 + 2x - 0.0012x^2$. Write down the revenue function $R(x)$ and the profit function $\Pi(x)$. Then, find maximum profit.

$R(x) = x \cdot P(x) = -0.0032x^2 + 10x$

$\Pi(x) = R(x) - C(x) = -0.0032x^2 + 10x - 4000 - 2x + 0.0012x^2$

$= -0.002x^2 + 8x - 4000$

$\Pi'(x) = -0.004x + 8$. $\Pi'(x) > 0$ for $x \in (0, 2000)$ and $\Pi'(x) < 0$ for $x > 2000$.

Hence $\Pi(2000)$ is the maximum profit.

$\Pi(2000) = 4000$.